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Cardinal Splines of Odd Degree on Uniform Meshes*

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Consider a polynomial spline $C(t)$ of degree n in the class $C^{n-1}(-\infty, \infty)$ with its nodes at the integers and which satisfies

$$C(j) = \delta_{0j} \qquad (j = 0, \pm 1,...). \tag{1}$$

Such a spline is commonly referred to as a *cardinal* spline. If n is odd, then $C(t)$ exists provided the values of $C^{(n-1)}(t)$ at the nodal points satisfy the doubly-infinite system of linear equations

$$\begin{pmatrix} & & \cdots & & \\ \cdots\ C_{-1}(n) & C_0(n) & C_1(n) & C_2(n)\ \cdots \\ \cdots\ C_{-2}(n) & C_{-1}(n) & C_0(n) & C_1(n)\ \cdots \\ \cdots\ C_{-3}(n) & C_{-2}(n) & C_{-1}(n) & C_0(n)\ \cdots \\ & & \cdots & & \end{pmatrix} \begin{bmatrix} \vdots \\ M_{-2} \\ M_{-1} \\ M_0 \\ M_1 \\ M_2 \\ \vdots \end{bmatrix} = n! \begin{bmatrix} \vdots \\ -\binom{2\bar{n}}{\bar{n}-2} \\ \binom{2\bar{n}}{\bar{n}-1} \\ -\binom{2\bar{n}}{\bar{n}} \\ \binom{2\bar{n}}{\bar{n}+1} \\ -\binom{2\bar{n}}{\bar{n}+2} \\ \vdots \end{bmatrix} (-1)^{\bar{n}+1}, \tag{2}$$

where $M_j = C^{(n-1)}(j)$ $(j = 0, \pm 1,...)$, the matrix entries $C_j(n)$ are given in [1], $\bar{n}$ is defined by $2\bar{n} = n - 1$,[1] and we let $\binom{2\bar{n}}{k} = 0$ for $k < 0$ or $k > 2\bar{n}$. Under

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[1] We assume throughout this discussion that $\bar{n} > 0$. The case $\bar{n} = 0$ is the piecewise linear case and offers no difficulty. For an early investigation of the interpolation problem defined by (1) and more general problems, see [4].

these circumstances the quantities M_j can be given explicitly and no other spline satisfying (1) has a bounded $(n-1)$-th derivative. These assertions are an immediate consequence of the analysis contained in [1].

Assuming for the moment that we have solved the system (2) for the quantities M_j, we then could exhibit $C(t)$ explicitly by proceeding as follows: Consider the related spline $\hat{C}(t)$ defined on $[0, \infty)$ by

$$\hat{C}(t) = \frac{1}{n!}(M_1 - M_0)\, t^n \qquad (0 \leqslant t \leqslant 1),$$

$$\hat{C}(t) = \frac{1}{n!}(M_1 - M_0)\, t^n + \frac{1}{n!}(M_2 - 2M_1 + M_0)(t-1)^n \qquad (1 \leqslant t \leqslant 2),$$

$$\vdots \qquad\qquad \vdots$$

$$\hat{C}(t) = \frac{1}{n!}(M_1 - M_0)\, t^n + \frac{1}{n!}\sum_{k=1}^{j}(M_{k+1} - 2M_k + M_{k-1})(t-k)^n \qquad (j \leqslant t \leqslant j+1),$$

$$\vdots \qquad\qquad \vdots$$

We now extend the definition of $\hat{C}(t)$ to $(-\infty, \infty)$ by letting $\hat{C}(t) = \hat{C}(-t))$ for $t \leqslant 0$. Our definition of $\hat{C}(t)$ is such that

$$\hat{C}^{(n)}(t) = C^{(n)}(t) \qquad (-\infty < t < \infty) \tag{4}$$

or equivalently,

$$C(t) = \hat{C}(t) + P_\infty(t) \qquad (-\infty < t < \infty), \tag{5}$$

where $P_\infty(t)$ is a polynomial of degree $n-1$. In asserting that (4) is valid on $(-\infty, \infty)$ and not simply $[0, \infty)$ we are using the fact that the matrix in (2) is symmetric about the diagonal containing the entries $C_0(n)$ and the right hand member is symmetric about the entry $-\binom{2n}{n}$.

A straightforward procedure would be to determine $P_\infty(t)$ from n interpolation conditions such as

$$P_\infty(j) = \delta_{0j} - \hat{C}(j) \qquad (j = 0, 1,..., n-1). \tag{6}$$

This, in fact, could be done with the result that the formula

$$C(t) = \frac{1}{n!}(M_1 - M_0)\, t^n + \frac{1}{n!}\sum_{K=1}^{j}(M_{K+1} - 2M_K + M_{K-1})(t-K)^N + P_\infty(t) \tag{7}$$

would be completely determined and would define $C(t)$ on the interval $[j, j+1]$ for any nonnegative integer j. The formula $C(t) = C(-t)$ would then define $C(t)$ on the corresponding interval $(-j-1, -j)$.

Numerically, this is not eery satisfactory for large values of j since the evaluation of the summation is time-consuming and inherently suffers from rounding errors. Consequently, we modify our approach so as to obtain

$C(t)$ in a more compact form which sheds considerable light on its intrinsic structure.

In the cubic case it is known [2, 3] that

$$\begin{aligned} C(t) &= (3\lambda + 2)\, t^3 - 3(\lambda + 1)\, t^2 + 1 \qquad (0 \leqslant t \leqslant 1), \\ C(t) &= 3\lambda^j[(\lambda + 1)(t - j)^3 - (\lambda + 2)(t - j)^2 + (t - j)] \\ &\qquad (j \leqslant t \leqslant j + 1; \quad j = 1, 2, \ldots), \end{aligned} \tag{8}$$

where $\lambda = -2 + \sqrt{3}$. Since $C(t)$ is an even function, the relation $C(t) = C(-t)$ defines $C(t)$ on $(-\infty, 0)$. It follows from (8) that essentially only two cubic arcs are needed to define $C(t)$ on $[0, \infty)$ and hence on $(-\infty, \infty)$: one for $[0, 1]$ and one for $[1, 2]$. The arc for $[j, j + 1]$ differs from the arc for $[1, 2]$ only in that $t - 1$ is replaced by $t - j$ and arc equation is multiplied by λ^{j-1}. We now seek to obtain the analogue of this result for higher odd values of n.

We have already utilized the auxiliary spline $\hat{C}(t)$ for which we have on $[0, \infty)$ the representation

$$\hat{C}(t) = \frac{1}{n!}(M_1 - M_0)\, t^n + \frac{1}{n!}\sum_{j=K}^{j} (M_{K+1} - 2M_K + M_{K-1})(t - K)^n \qquad (0 \leqslant j \leqslant t \leqslant j + 1). \tag{9}$$

Expanding $(t - K)^n$ by the binomial theorem and interchanging the order of summation we obtain

$$\hat{C}(t) = \sum_{j=0}^{n} \omega_{lj} t^l \qquad (0 \leqslant j \leqslant t \leqslant j + 1), \tag{10}$$

where

$$\begin{aligned} \omega_{lj} &= \frac{1}{n!}\binom{n}{l} \sum_{K=1}^{j} (-K)^{n-l}(M_{K+1} - 2M_K + M_{K-1}) \qquad (l = 0, 1, \ldots, n - 1), \\ \omega_{nj} &= \frac{1}{n!}\Big(M_1 - M_0 + \sum_{\substack{K=1 \\ j \geqslant K}}^{j} (M_{K+1} - 2M_K + M_{K-1})\Big) = \frac{1}{n!}(M_{j+1} - M_j). \end{aligned} \tag{11}$$

Using Eq. (5) and the fact [1] that

$$M_K = O(|\, r_{\bar{n}}\,|^K), \tag{12}$$

as $K \to \infty$ where $-1 < r_{\bar{n}} < 0$, it follows that $\lim_{j\to\infty} \omega_{lj}$ exists for $l = 0, 1, \ldots, n$. Moreover, since $C(j) = 0$ for $j > 0$, the growth of the nonconstant terms of $\hat{C}(t)$ must be offset by that of the nonconstant terms of $P_\infty(t)$ as t becomes infinite. As a consequence, if we let

$$P_\infty(t) = \sum_{l=0}^{n} \omega_l t^l, \tag{13}$$

then

$$\begin{aligned}\omega_l &= -\lim_{j\to\infty} \omega_{lj} \\ &= -\frac{1}{n!}\binom{n}{l}\sum_{K=1}^{\infty}(-K)^{n-l}(M_{K+1}-2M_K+M_{K-1})\end{aligned} \tag{14}$$

for $l = 1, 2,..., n$. In particular,

$$\omega_{n-1} = -\lim_{j\to\infty} \omega_{n-1,j} = \frac{1}{(n-1)!} M_0 \,. \tag{15}$$

Also, we observe that

$$\omega_n = -\lim_{j\to\infty} \omega_{nj} = 0; \tag{16}$$

thus Eq. (16) is consistent with the fact that from earlier considerations $P_\infty(t)$ is known to be of degree $n - 1$. Furthermore, since $C(0) = 1$, the constant term in $P_\infty(t)$ must be unity.

In view of the preceding discussion we are led to the representation

$$\begin{aligned}C(t) &= \frac{1}{n!}(M_1 - M_0)\, t^n + \frac{1}{(n-1)!} M_0 t^{n-1} \\ &\quad - \frac{1}{n!}\sum_{l=1}^{n-2}\binom{n}{l}\sum_{K=1}^{\infty}(-K)^{n-l}(M_{K+1}-2M_K+M_{K-1})\, t^l + 1 \\ &\qquad\qquad (0 \leqslant t \leqslant 1).\end{aligned} \tag{17}$$

For $0 < j \leqslant t \leqslant j + 1$ we have

$$\begin{aligned}C(t) &= \frac{1}{n!}(M_{j+1} - M_j)\, t^n \\ &\quad - \frac{1}{n!}\sum_{l=1}^{n-1}\binom{n}{l}\sum_{K=j+1}^{\infty}(-K)^{n-l}(M_{K+1}-2M_K+M_{K-1})\, t^l + \text{const} \\ &= \frac{1}{n!}(M_{j+1} - M_j)\, t^n \\ &\quad - \frac{1}{n!}\sum_{K+j=1}^{\infty}(M_{K+1}-2M_K+M_{K-1})\sum_{l=1}^{n-1}\binom{n}{l}(-K)^{n-l}t^l + \text{const} \\ &= \frac{1}{n!}\left[M_{j+1} - M_j + \sum_{K=j+1}^{\infty}(M_{K+1}-2M_K+M_{K-1})\right] t^n \\ &\quad - \frac{1}{n!}\sum_{K=j+1}^{\infty}(M_{K=1}-2M_K+M_{K-1})\sum_{l=0}^{n}\binom{n}{l}(-K)^{n-l}t^l + \text{const} \\ &= -\frac{1}{n!}\sum_{K=j+1}^{\infty}(M_{K+1}-2M_K+M_{K-1})(t-K)^n + \text{const}\end{aligned}$$

We now express $C(t)$ in powers of $t-j$. Thus, the constant term vanishes and

$$C(t) = -\frac{1}{n!}\sum_{K=j+1}^{\infty}(M_{K+1}-2M_K+M_{K-1})(t-j-(K-j))^n + \text{const}$$

$$= -\frac{1}{n!}\sum_{K=j+1}^{\infty}(M_{K+1}-2M_K+M_{K-1})\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}(K-j)^{n-l}(t-j)^l + \text{const}$$

$$= -\frac{1}{n!}\sum_{l=1}^{n}(-1)^{n-l}\binom{n}{l}\left[\sum_{K=j+1}^{\infty}(M_{K+1}-2M_K+M_{K-1})(K-j)^{n-l}\right](t-j)^l.$$

Consequently, we have

$$C(t) = \sum_{l=1}^{n}\alpha_{jl}(t-j)^l \qquad (0<j\leqslant t\leqslant j+1), \tag{18}$$

where

$$\alpha_{jl} = (-1)^{n-l+1}\frac{1}{n!}\binom{n}{l}\sum_{K=j+1}^{\infty}(M_{K+1}-2M_K+M_{K-1})(K-j)^{n-l}. \tag{19}$$

In view of (12) the sums in (19) are convergent so that the coefficients α_{jl} $(l=1, 2,..., n)$ are well-defined.[2]

From the analysis in [1] it follows that

$$\begin{aligned} M_{-K} = M_K &= n!\sum_{j=-\bar{n}}^{\bar{n}}(-1)^{j+1}\binom{2\bar{n}}{\bar{n}+j}a_{j+K} \\ &= n!\sum_{j=-\bar{n}}^{\bar{n}}(-1)^{j+1}\binom{2\bar{n}}{\bar{n}+k}\sum_{\omega=1}^{\bar{n}}a_{\omega,j+K} \qquad (K=0, 1, 2,...), \end{aligned} \tag{20}$$

where

$$a_{\omega,j+K} = \frac{r_\omega^{\bar{n}+|j+K|}}{P_n'(r_{\omega,1})} \qquad (\omega = 1, 2,..., \bar{n}). \tag{21}$$

Here each r_ω is a nonzero root of the Hille polynomial $P_n(z, 1)$ interior to the unit interval. With no loss in generality we assume $r_{\bar{n}} < r_{\bar{n}-1} < \cdots < r_1$. Now let

$$M_K(\omega) = n!\sum_{j=-\bar{n}}^{\bar{n}}(-1)^{j+1}\binom{2\bar{n}}{\bar{n}+j}\frac{r_\omega^{\bar{n}+|j+K|}}{P_n'(r_{\omega,1})} \tag{22}$$

[2] Since the quantities M_j have not been specified up to this point, Eqs. (17)–(19) are applicable to a much larger class of interpolation problems on $(-\infty, \infty)$ than that defined by (1).

and let $C_\omega(t)$ be the spline defined by (17)–(19) with M_j replaced by $M_j(\omega)$ except that on the interval [0, 1] we replace the constant term by $1/\bar{n}$. Thus,

$$C(t) = \sum_{\omega=1}^{n} C_\omega(t). \tag{23}$$

It should be noted that in the sense that

$$C_\omega(j) = \frac{1}{n}\,\delta_{0j}\,,$$

the splines $C_\omega(t)$ resemble cardinal splines. However, except for $\bar{n} = 1$, they are not in $C^{n-1}(-\infty, \infty)$.

Suppose, now, that $K > \bar{n}$. Then,

$$\begin{aligned}
M_K(\omega) &= n!\sum_{j=-\bar{n}}^{\bar{n}} (-1)^{j+1}\binom{2\bar{n}}{\bar{n}+j}\frac{(r_\omega)^{\bar{n}+j+K}}{P_n'(n, 1)} \\
&= r_\omega n!\sum_{j=-\bar{n}}^{\bar{n}} (-1)^{j+1}\binom{2\bar{n}}{\bar{n}+j}\frac{(r_\omega)^{\bar{n}+j+K-1}}{P_n'(r_\omega\,, 1)} \\
&= r_\omega M_{K-1}(\omega).
\end{aligned} \tag{24}$$

Consequently, for $j > \bar{n}$, we have

$$\begin{aligned}
\alpha_{jl}(\omega) &= -\frac{1}{n!}\binom{n}{l}\sum_{K=j+1}^{\infty} [M_{K+1}(\omega) - 2M_K(\omega) + M_{K-1}(\omega)](K-j)^{n-l} \\
&= -\frac{r_\omega}{n!}\binom{n}{l}\sum_{K=j+1}^{\infty} [M_K(\omega) - 2M_{K-1}(\omega) + M_{K-2}(\omega)](K-j)^{n-l} \\
&= -\frac{r_\omega}{n!}\binom{n}{l}\sum_{K=j}^{\infty} [M_{K+1}(\omega) - 2M_K(\omega) + M_{K-1}(\omega)](K-(j-1))^{n-l} \\
&= r_\omega \alpha_{j-1,l}(\omega) \qquad (l = 0, 1, 2,..., n).
\end{aligned} \tag{25}$$

Thus, since the constant term $\alpha_{j0}(\omega)$ vanishes when the index $j \geqslant \bar{n}$, we see for $j > \bar{n}$ that the arc of $C_\omega(t)$ on the interval $j \leqslant t \leqslant j + 1$ is the same as the arc of $C_\omega(t)$ on the interval $j - 1 \leqslant t \leqslant j$ except that $t - j + 1$ is replaced by $t - j$ and the coefficients are multiplied by r_ω. It follows that each $C_\omega(t)$ behaves in the manner previously observed for $C(t)$ itself in the cubic case except that there are $\bar{n} + 1$ arcs instead of two arcs.

Let us give $P_n(z, t)$ the representation

$$P_n(z, 1) = z(b_{-\bar{n}}z^{n-1} + b_{-\bar{n}+1}z^{n-2} + \cdots + b_0 z^{\bar{n}} + \cdots + b_{\bar{n}}). \tag{26}$$

Consider the spline

$$B_n(t) = \sum_{l=-\bar{n}}^{\bar{n}} b_l C(t-l). \tag{27}$$

If $j \geqslant 2\bar{n}$, then on the interval $j \leqslant t \leqslant j+1$ we have

$$\begin{aligned} B_n(t) &= \sum_{l=-\bar{n}}^{\bar{n}} b_l \sum_{\omega=1}^{\bar{n}} \sum_{s=1}^{n} \alpha_{j-l,s}(\omega)(t-j)^s \\ &= \sum_{l=-\bar{n}}^{\bar{n}} b_l \sum_{\omega=1}^{\bar{n}} (r_\omega)^{-l} \sum_{s=1}^{n} \alpha_{js}(\omega)(t-j)^s \\ &= \sum_{\omega=1}^{\bar{n}} \left\{ \sum_{s=1}^{n} \alpha_{js}(\omega)(t-j)^s \right\} \left\{ \sum_{l=-\bar{n}}^{\bar{n}} b_l (r_\omega)^l \right\} \\ &= \sum_{\omega=1}^{\bar{n}} \frac{1}{(r_\omega)^{n+1}} \left\{ \sum_{s=1}^{n} \alpha_{js}(\omega)(t-j)^s \right\} P_n(r_\omega, 1) \\ &= 0, \end{aligned} \tag{28}$$

Thus, the spline $B_n(t)$ vanishes identically outside a region consisting of $2\bar{n}$ intervals on each side of $t = 0$.

In fact, aside from a constant factor the splines $B_n(t)$ are included among the splines with compact support considered by Schoenberg [4, 5]. Thus, the support of the splines is actually smaller than the preceding argument indicates. This is easily seen from the analysis contained in [4] and [6].

We close this paper with an example: we construct the cubic cardinal spline $C(t)$ using our formulas and verify that the resulting equations are, indeed, in agreement with Eq. (8).

In the cubic case we have $\bar{n} = 1$ so that M_0 and M_1 are the only M_j that need be calculated. Let r_1 be denoted by λ. Then from (2)

$$M_0 = 3! \sum_{j=-1}^{1} (-1)^{j+1} \binom{2}{1+j} \frac{\lambda^{1+|j+0|}}{P_3'(\lambda, 1)}.$$

But

$$P_3(z, 1) = z(z^2 + 4z + 1),$$

so that

$$P_3'(z, 1) = (z^2 + 4z + 1) + z(2z + 4).$$

Consequently,

$$P_3'(\lambda, 1) = \lambda^2 - 1,$$

since $\lambda^2 + 4\lambda + 1 = 0$. Thus,

$$\begin{aligned} M_0 &= \frac{6}{\lambda^2 - 1} [\lambda^2 - 2\lambda + \lambda^2] \\ &= \frac{12\lambda}{\lambda + 1}. \end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned} M_1 &= \frac{6}{\lambda^2 - 1} \sum_{j=-1}^{1} (-1)^{j+1} \binom{2}{1+j} \lambda^{2+j} \\ &= \frac{6}{\lambda^2 - 1} [\lambda - 2\lambda^2 + \lambda^3] \\ &= \frac{6\lambda(\lambda - 1)}{\lambda + 1}. \end{aligned} \tag{30}$$

On the interval $0 \leqslant t \leqslant 1$ we have

$$\begin{aligned} C(t) &= \frac{1}{3!} (M_1 - M_0)\, t^3 + \frac{1}{2} M_0 t^2 - \frac{1}{3!} \binom{3}{1} \sum_{K=1}^{\infty} K^2 (M_{K+1} - 2M_K + M_{K-1}) t + 1 \\ &= \frac{1}{6} \left(\frac{1}{2} (\lambda - 1) - 1\right) \frac{12\lambda}{\lambda + 1}\, t^3 + \frac{6\lambda}{\lambda + 1}\, t^2 \\ &\quad - \tfrac{1}{2}[M_0 - 2M_1 + M_2 + 4M_1 - 8M_2 + 4M_3 + 9M_2 - 18M_3 + 9M_4 \\ &\quad + 16M_3 + 32M_4 + \cdots]\, t + 1 \\ &= \frac{(\lambda - 3)\lambda}{\lambda + 1}\, t^3 + \frac{6\lambda}{\lambda + 1}\, t^2 - \left[\frac{1}{2} M_0 + M_1(1 + \lambda + \lambda^2 + \cdots)\right] t + 1 \\ &= \frac{(\lambda - 3)\lambda}{\lambda + 1}\, t^3 + \frac{6\lambda}{\lambda + 1}\, t^2 + \left(\frac{1}{2} + \frac{1}{2} (\lambda - 1) \frac{1}{1 - \lambda}\right) M_0 t + 1. \end{aligned}$$

Consequently,

$$C(t) = (3\lambda + 2)\, t^3 - 3(\lambda + 1)\, t^2 + 1, \tag{31}$$

if we take into account the relation

$$\lambda^2 + 4\lambda + 1 = 0.$$

Finally, on an interval $0 < j \leqslant t \leqslant j + 1$, we have

$$C(t) = \frac{1}{3!} \sum_{l=1}^{3} (-1)^{3-l+1} \binom{3}{l} \sum_{K=j+1}^{\infty} (M_{K+1} - 2M_K + M_{K-1})(K-j)^{3-l}(t-j)^l$$

$$= -\frac{1}{6} \times 3 \sum_{K=j+1}^{\infty} (M_{K+1} - 2M_K + M_{K-1})(K-j)^2(t-j)$$

$$+ \frac{1}{6} \times 3 \sum_{K=j+1}^{\infty} (M_{K+1} - 2M_K + M_{K-1})(K-j)(t-j)^2$$

$$- \frac{1}{6} \sum_{K=j+1}^{\infty} (M_{K+1} - 2M_K + M_{K-1})(t-j)^3$$

$$= -\frac{1}{2} M_1 \sum_{K=j+1}^{\infty} (\lambda^2 - 2\lambda + 1)\, \lambda^{K-2}(K-j)^2(t-j)$$

$$+ \frac{1}{2} M_1 \sum_{K=j+1}^{\infty} (\lambda^2 - 2\lambda + 1)\, \lambda^{K-2}(K-j)(t-j)^2$$

$$- \frac{1}{6} M_1 \sum_{K=j+1}^{\infty} (\lambda^2 - 2\lambda + 1)\, \lambda^{K-2}(t-j)^3$$

$$= -\frac{M_1}{2} (\lambda - 1)^2 \lambda^{j-1} \sum_{K=0}^{\infty} K^2 \lambda^{K-1}(t-j)$$

$$+ \frac{M_1}{2} (\lambda - 1)^2 \lambda^{j-1} \sum_{K=0}^{\infty} K \lambda^{K-1}(t-j)^2$$

$$- \frac{1}{6} M_1(\lambda - 1)^2 \lambda^{j-1} \sum_{K=0}^{\infty} \lambda^K (t-j)^3$$

$$= -\frac{M_1}{6} (\lambda - 1)^2 \lambda^{j-1} \left[(t-j)^3 \sum_{K=0}^{\infty} \lambda^K - 3(t-j)^2 \sum_{K=0}^{\infty} K\lambda^{K-1} \right.$$

$$\left. + 3(t-j) \sum_{K=0}^{\infty} K^2 \lambda^K \right]$$

$$= -\frac{(\lambda - 1)^3 \lambda^j}{\lambda + 1} \left[\frac{1}{1-\lambda} (t-j)^3 - \frac{3}{(1-\lambda)^2} (t-j)^2 \right.$$

$$\left. + \frac{3(\lambda + 1)}{(1-\lambda)^3} (t-j) \right]$$

$$= \frac{(1-\lambda)^2}{\lambda + 1} \lambda^j (t-j)^3 - \frac{3(1-\lambda)}{\lambda + 1} \lambda^j (t-j)^2 + 3\lambda^j (t-j).$$

Again, since $\lambda^2 + 4\lambda + 1 = 0$, we obtain

$$C(t) = 3\lambda^j[(\lambda + 1)(t - j)^3 - (\lambda + 2)(t - j)^2 + (t - j)]. \tag{32}$$

Since Eqs. (29) and (30) are identical with Eqs. (8) our formulas are verified for the cubic case. For $\bar{n} > 1$ no great algebraic simplification takes place, but our formulas are still quite amenable to numerical computation for reasonable values of $\bar{n}$.

One useful application of cardinal splines on $(-\infty, \infty)$ is that of obtaining a basis of splines for a finite interval [a, b]. We can assume without loss in generality that $b - a$ is an integer since the modifications necessary for an arbitrary spacing h are minor. We now take the translations of $C(t)$ so that each such translation has its non-zero nodal point at one of the nodes in [a, b]. If this is done, and the resulting splines are restricted to [a, b], then —neglecting end conditions—we have the desired basis. To satisfy end conditions we add the restrictions of the translations centered at the $\bar{n}$ nodal points immediately to the left of $t = a$ and immediately to the right of $t = b$. Since all the basis splines are translates of the single spline $C(t)$ only one spline is really involved. Thus a considerable saving in computer storage requirements can be made.

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